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Backward-Euler Matrix Lyapunov Solver
Adaptive, always-valid covariance

In plain words: it predicts how strongly each floor of a building is likely to sway during an earthquake — as statistics, not thousands of simulations — and the answer is always physically meaningful.

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01 Random shaking, uncertain response

No two earthquakes are alike, so engineers need the statistics of a building's response — not one guess.

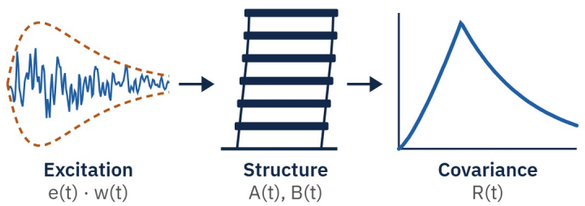
Earthquake, wind and traffic loads are random and **non-stationary**: their intensity builds up and dies away. Design checks need how the **second moments** of the response — variances and covariances of drift, velocity and acceleration — evolve in time.

For linear structures they follow from one matrix ODE — if it is solved accurately, stably and without breaking physics.

02 One equation, no simulations

Instead of running thousands of random simulations, follow how the spread of the motion evolves.

$$\dot{z} = A(t)\,z + G(t)\,w(t),\;\; E[w\,w^T] = J(t)\,\delta$$
$$\dot{R} = A(t)\,R + R\,A^T(t) + B(t),\;\; B = G\,J\,G^T$$



Schematic. R(t) is the state covariance: the spread of motion. Coloured excitation enters by adding its filter to the state.

03 No impossible answers, by design

A variance can never be negative. This scheme guarantees that at every step — by design.

- **Always valid.** If $R_n \succeq 0$ and $B \succeq 0$, then $R_{n+1} \succeq 0$ (positive semidefinite).
- **Stiff-proof.** A- and L-stable: fast higher modes cannot make it blow up.
- **A deliberate trade.** First order only — higher-order methods can return negative variances.
- **Matrix form, O(m³)**, not the O(m⁶) vectorized system.

04 Try it, then how it works

Describe the structure and the shaking; get the statistics back.

README · MINIMAL EXAMPLE

```
% SDOF: ω = 10 rad/s, 5 % damping
A   = [0 1; -100 -1];
g   = [0; 1];  S0 = 0.01;
% modulated white-noise intensity
phi = @(t) min(1,t/2).*exp(-0.08*max(t-12,0));
B   = @(t) 2*pi*S0*phi(t)^2*(g*g');
opts = lyap_be_options('RelTol',1e-4, ...
   'InitialStep',0.01,'MaxStep',0.5, ...
   'StateScale',[0.1; 1.0]);
sol = lyap_be_adaptive(A,B,[0 30], ...
      zeros(2),opts);
plot(sol.t, sqrt(squeeze(sol.R(1,1,:))))
```

Ten lines, one call: ≈ 2 s in GNU Octave, 1,874 adaptive steps.

INTEGRATE

Backward-Euler step
Implicit update of the covariance R over one step h.

One algebraic Lyapunov equation per step
Shifting A by −I/(2h) turns each update exactly into M R + R M^T + Q = 0.

Complex-Schur Bartels–Stewart solve
A standard, robust Lyapunov solver in base MATLAB.

CONTROL

Step doubling
One step h against two of h/2 gives the local error.

Accept, reject, resize
Keep if err ≤ 1, else retry smaller; a PI controller sizes the next step.

OUTPUT

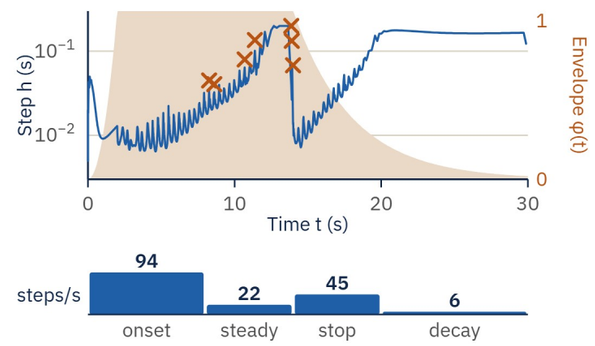
What comes back
R(t) at every step, step and error history, and a negative-variance check.

SCOPE AND LIMITS
Linear systems (nonlinear via equivalent linearization); first order; dense core, not for very large sparse models.

05 Small steps only when it matters

It slows down whenever the statistics change — at onset and when shaking stops — and strides through the steady part.

$$E = R^{[h/2,\,h/2]} - R^{[h]}\text{(Richardson, p = 1)}$$
$$err = \|(E \oslash S) \oslash W\|_F / m \le 1$$

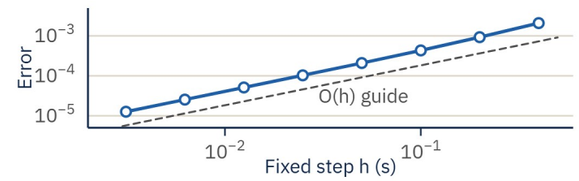


Six-storey example: 1,212 accepted and 7 rejected steps (×). Effort follows how fast the covariance changes, not how hard the ground shakes.

06 Proof that it's right

All four built-in tests pass (run_tests), against exact answers, the long-run limit and physics.

1 · Exact scalar solution



Error halves with h (first order). **2 · Stationary limit:** error 7.5 × 10^{-16}. **3 · Symmetry, PSD:** exact; $\lambda_{min}/\lambda_{max} \ge -3 \times 10^{-17}$.

4 · Tolerance response — six-storey model

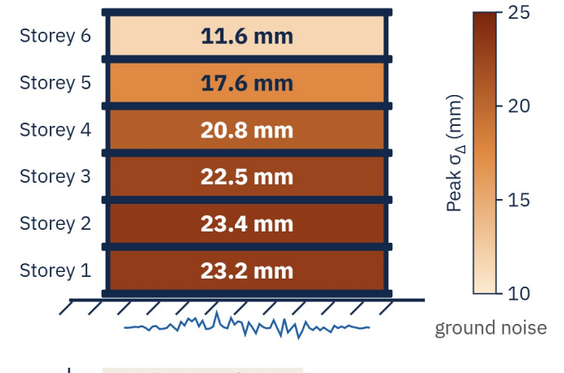
RelTol	Steps	Solves	Peak σ_{u1} (mm)
10^{-2}	307	954	23.223
10^{-3}	762	2,310	23.230
10^{-4}	1,787	5,379	23.239
3.0×10^{-5}	2,431	7,296	23.240

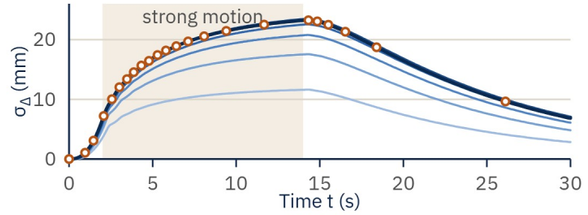
07 Where does the building deform?

Package example 04 · six-storey shear frame · T_1 = 1.39 s · 2 % damping

Storeys 1–2 take twice the interstorey drift of storey 6, even though the roof moves furthest.

Peak drift σ per storey, reached at t ≈ 14.2 s





Drift ≈ storey shear ÷ stiffness, largest at the base. **Circles**: independent reference, within 0.2 %.

08 Get the code

Free, open source (MIT), with a 28-page manual.

ARCHIVE · ZENODO
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SOURCE · GITHUB
ARtIStE-Research-Group/backward-Euler-Lyapunov-solver

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